Cell structure in waves diffracted by a wedge

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Abstract: Waves diffracted by a wedge that is made of perfectly reflecting material exhibit characteristic spatial behavior depending on an aperture angle of wedge. For examples in a perfectly reflecting plane and a corner cube diffracted waves are identically zero. And in a semi-infinite plane diffracted waves are symmetric with respect to the central axis of wedge. The relation between the behavior and the aperture angle, however, has not been studied in detail so far since there is no appreciate model for diffracted waves and the rigorous solution for diffracted waves is so complex that only the simplest case can be analyzed. We have proposed the new mathematical model for diffracted waves where they are expressed as a sum of two more fundamental quantities called elementary diffracted waves. Thus this provides fair means for analyzing the global property of waves diffracted by a wedge. The new model reveals that cell structure exists in waves diffracted by a wedge. The cases mentioned above can be explained in terms of the cell structure.

Keywords: Diffraction, Wedge, Cell

1. Introduction

We have proposed a new physical principle that is called virtual discontinuity principle of diffraction (abbreviated VDPD) for analyzing waves diffracted by perfectly reflecting objects and formulated a mathematical model for calculating diffracted waves in terms of more fundamental quantity called elementary diffracted waves 1,2. The special merit of our model lies in the fact that diffracted waves calculated by the model satisfy the boundary condition at the surface of the object inherently. This property is not supported by other principles for analyzing diffracted waves, for examples, Kirchhoff’s formula, geometrical theory of diffraction, and Boundary Element Method. Furthermore the simple and understandable definition of elementary diffracted waves makes the analysis clear and straightforward and in the previous report the high frequency approximate solution of waves diffracted by a wedge can be derived by means of the new model where it has been difficult to reduce that from the rigorous solution 3.

In this paper the analysis by elementary diffracted waves casts a new light on the global property of waves diffracted by a wedge. Waves diffracted by a wedge made of perfectly reflecting material exhibit the global property at some aperture angles of wedge. For examples, diffracted waves are identically zero in a wedge of aperture angle $\theta$, that is, a perfectly reflecting plane and that of aperture angle $\theta/2$, that is, a corner cube. And in a wedge of aperture angle $2\theta$, that is, a semi-infinite plane, diffracted waves are symmetric with respect to the central axis of wedge. The relation between the behavior and the aperture angle, however, has not been studied in detail so far since there is no appreciate model for diffracted waves and the rigorous solution for diffracted waves is so complex that only the simplest case can be analyzed. The new mathematical model by VDPD, however, reveals that cell structure exists in waves diffracted by a wedge.

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wedge. If the aperture angle normalized by $\Theta$ can be expressed by $N/M$ where $N$ and $M$ are relatively prime integers, the wedge region can be divided equally into $N$ cells and diffracted waves in a cell can be reconstructed by diffracted waves in $N-I$ cells remained. The perfectly reflecting plane and the corner cube correspond to the case of $N=1$ and the semi-infinite plane to the case of $N=2$. In these cases the cell structure is visually recognizable but in most cases it isn’t. Consequently it has never been noticed that the universal cell structure exists in waves diffracted by the wedge.

This paper is organized as follows. In Sec.2 wedge-shaped region, potential in the region and elementary diffracted waves are mathematically defined. In Sec.3 a virtual space is formulated by incorporating mirror images reflected by edges of wedge and the Green’s theorem is applied to the virtual space to formulate a mathematical model for diffracted waves in terms of elementary diffracted waves. In Sec.4 the cell structure in diffracted waves is derived from the model and some examples of the cell structure are illustrated graphically. Short summary is given in Sec.5.

![Figure 1 Wedge-shaped region](image)

2. Elementary diffracted waves

Draw a half line in the two-dimensional space and denote the starting point as $Q$ and introduce the polar coordinate system $r=(r, \theta)$ by specifying $r$ as a distance measured from $Q$ and $\theta$ as an angle measured from the half line in the anticlockwise direction. Let $W_0$ be a wedge-shaped region (abbreviated wedge) of aperture angle $\Theta$ where $\Theta=2\pi$ corresponds to an semi-infinite plane, $\Theta>\pi$ a concave wedge, $\Theta<\pi$ a convex wedge and $\Theta=\pi$ a reflecting plane. The apex of $W_0$ lies on $Q$ and $W_0$ is bounded by two edges $B_0^a$ and $B_0^c$. Let $L(\theta)$ denote a half line that starts from $Q$ and runs in the $\theta$ direction. Then the edges can be specified as $B_0^a=L(\pi/2)$ and $B_0^c=L(-\pi/2)$ as shown in Figure 1.

The waves propagating in $W_0$ are stationary in time and satisfy the following relation

$$\nabla^2 U + k^2 U = -\delta(r - r_S)$$

where $U$ stands for the potential of waves, $k=2\pi / \lambda$ the wave number ($\lambda$ : the wavelength), $\delta$ the delta function, $r_S=(r, \theta)$ the position vector of the point source $S$. In the free space $S$ radiates the direct waves $U^F$ that is given by

$$U^F(r | r_S) = H_0^{(2)}(k | r - r_S |)/(4\pi)$$

where $H_0^{(2)}$ stands for the 0-th order Hankel function of the second kind, $j$ the imaginary unit and the stationary time function $\exp(j\omega t)$ is deleted where $\omega$ stands for the angular frequency. As to the boundary condition, the Dirichlet condition ($\partial U/\partial n = 0$) or the Neumann condition ($U = 0$) is set to edges of the wedge where $n$ stands for an inner unit vector normal to the edges. Let denote the wedge that satisfies the Dirichlet condition as the hard wedge and the Neumann condition as the soft wedge. The edge of the hard wedge can be regarded as a mirror of $m = 1$ where $m$ stands for the reflection coefficient of the mirror.
Similarly the edge of the soft wedge can be regarded as a mirror of \( m = -1 \). The diffracted waves can be considered as a deviation from the geometrical optics waves. Then the potential can be expressed as

\[
U(r) = U^G (r) + U^D (r),
\]

where \( U^G \) stands for the potential for the geometrical optics waves and \( U^D \) that for the diffracted waves.

In this paper a new field quantity that is called elementary diffracted waves is introduced by the following relation

\[
E(r, \theta) = \int_{L(\theta)} -g(k(r + \ell))\partial U(r_i)/\partial n_i \, d\ell,
\]

where \( E \) stands for the potential for the elementary diffracted waves, \( \ell \) the coordinate taken along \( L(\theta) \), \( r_i \) the position vector of a point on \( L(\theta) \), \( n_i \) an unit vector normal to \( L(\theta) \) taken in the anticlockwise direction as shown in Figure 1, and \( g \) the Green’s function that is given by

\[
g(x) = H_0^{(2)}(x)/(4j).
\]

As seen in (4) \( E \) can be calculated using \( U \) in \( W_0 \). Physically, however, \( E \) is considered as a contribution of \( U \) on \( L(\theta) \) to the point \( O \) that is located at \((r, \Phi + n)\) as shown in Figure 1. But in this case \( O \) is located in the area outside \( W_0 \). Inversely if \( O \) is kept in \( W_0 \), it might be necessary to draw \( L(\theta) \) in the area outside \( W_0 \). Thus it is necessary to make the structure of the space compatible to the physical concept of \( E \).

Figure 2 (a) Extension of wedge-shaped region, (b) Graph for wedge number, (c) Graph for original angle.

3. New model for diffracted waves

Extend the wedge-shaped region beyond the edges by considering the edges \( B_0^a \) and \( B_0^c \) as mirrors and mirrored images are assumed to be spread out beyond the edges. The wedges \( W_i \), \( i = 1,2,3,\ldots \) are spread out beyond \( B_0^a \) and the wedges \( W_i \), \( i = -1,-2,-3,\ldots \) are spread out beyond \( B_0^c \) as shown in Figure 2(a) where \( i \) stands for a number of wedge and \( |i| \) corresponds to the reflection number. The wedge \( W_i \) is bounded by \( L((i+1)\Phi/2) \) and \( L((i-1)\Phi/2) \) and let us denote the former as \( B_i^a \) and the latter as \( B_i^c \). If a point in \( W_0 \) that is specified by \( (r, \theta^*) \) is imaged to a point \( (r, \theta) \) in \( W_i \) by mirror reflections, then the following relation holds.

\[
\theta = (-1)^i \theta^* + i\Phi
\]

Inversely a point \( (r, \theta) \) in the extended wedge region is considered to belong to \( W_i \) and to be an image of the point \( (r, \theta^*) \) in \( W_0 \). Then the following relations hold

\[
i = \text{nint} (\theta/\Phi)
\]

\[
\theta^*/\Phi = (-1)^i (\theta/\Phi - i)
\]

where \( \text{nint}(x) \) gives the nearest integer of a real number \( x \) as shown in Figure 2(b) and \( y = \theta^*/\Phi \) is shown
in Figure 2(c) as a function of $x = \theta/\emptyset$. Let denote $\theta^*$ as the original angle of $\theta$ and it can be calculated by the triangular curve $y = f(x)$ in Figure 2(c). Then the potential at $(r, \theta)$ can be assigned by the following relation

$$U(r, \theta) = m^i U(r, \theta^*),$$

(9)

where $i$ and $\theta^*$ are calculated by (7) and (8) and $m^i$ reflects the amplitude reversal in the case of soft wedge $(m = -1)$. Similarly the following relation holds for the elementally diffracted waves

$$E(r, \theta) = (-m^i E(r, \theta^*).$$

(10)

Thus $U$ and $E$ can be extended beyond the edges continuously up to the first derivative. Note that these potentials are periodic in the extended wedge region and the following relations hold

$$U(r, \theta) = U(r, \theta + 2\Phi),$$

(11)

$$E(r, \theta) = E(r, \theta + 2\Phi).$$

(12)

Figure 3 Virtual space

A virtual space $V$ is defined as a space that can be observed geometrically by $O$ that is placed at $r = (r, \emptyset)$ in $W_0$ where the extended wedge region is assumed. It is assumed in $VDPD$ that any point in the sound field radiates secondary wavelet and it propagates like a particle. Thus the secondary wavelet that is radiated from any point in $V$ passes through $O$. Let introduce a half line $D = L(\emptyset + \emptyset)$, that is, a half line that starts from $Q$ and runs in the direction of $\emptyset + \emptyset$ as shown in Figure 3. It is a half line to calculate $E(r, \emptyset + \emptyset)$ in (4). If a point on $O$ is rotated in the anticlockwise direction until it touches $D$ in $W_p$ where $p = \text{nint}((\emptyset + \emptyset)\emptyset)$ is a nonnegative integer. Let $D^p$ express $D$ that is running in $W_p$ and $D_p^c$ a truncated wedge that is bounded by $B_p^c$ and $D^p$. Similarly if a point on $O$ is rotated in the clockwise direction, it touches $D$ in $W_n$ where $n = \text{nint}((\emptyset + \emptyset)\emptyset)$ is a nonpositive integer. Let $D^p$ express $D$ that is running in $W_n$ and $W_n^D$ a truncated wedge that is bounded by $B_n^a$ and $D^p$. Then the virtual space $V$ is formulated by

$$V = W_p^D \cup W_n^D \cup \left( \sum_{j=a+1}^{p-1} \cup W_j \right).$$

(13)

If either $p = 0$ or $n = 0$ holds, the third term in the right side of (13) becomes zero. The potentials on $D^p$ and $D^p$ are different for most cases and the potential in $V$ is not continuous at $D$. Accordingly it is called as a virtual discontinuity line.

Draw closed curves $C_i (i = n, 0, \ldots, p)$ in wedges that comprise $V$ in (13). Each $C_i$ is composed by two
circular arcs of radius $\bar{a}$ (<< $\bar{a}$) and $\bar{a}$ (>> $\bar{a}$) respectively and two segments connecting these arcs along the edges of $D$ as shown in Figure 3. The centers of curvature of arcs lie on $Q$ and the radii $\bar{a}$ and $\bar{a}$ are common for all $C_i$. A circle of very small radius centered at $O$ is included in $C_0$ and that centered at $S_i$ in $C_i$ where $S_i$ is the mirror image of $S$ in $W_i$ and $S_0 = S$. If $S_n$ is not included in $W_n$, no circle is added to $C_n$. The same story holds for $S_p$ and $C_p$. Then the following relation is obtained by applying the Green’s theorem to each $C_i$ and taking a sum of the resulting relations

$$U(r, \theta) = U^G(r, \theta) - E(r, \theta + \pi) + E(r, \theta - \pi).$$

(14)

Since the potential is continuous at edges, these pairs of integrals are cancelled out each other. And there remain the integrals along $D^a$ and $D^c$ since $D$ is the only edge in Figure 3 where the potential is discontinuous. In this case $O$ lies on the extension of $D$. Thus at the limits of $\bar{a}$ $\bar{a}$ $0$ and $\bar{a}$ $\bar{a}$, the integrals along $D^a$ and $D^c$ can be expressed by $-E(r; \bar{a} + \bar{a})$ and $E(r; \bar{a} - \bar{a})$ respectively and the integrals along the arcs become zeros. The geometrical optics solution $U^G$ in (14) is comprised by the sum of contributions from $S_i$. Then the potential for diffracted waves is expressed as

$$U^D(r, \theta) = -E(r, \theta + \pi) + E(r, \theta - \pi).$$

(15)

And in terms of the original angles (15) is rewritten as

$$U^D(r, \theta) = -(-m)^p E(r, (\theta + \pi)^*) + (-m)^q E(r, (\theta - \pi)^*)$$

(16)

where (15) and (16) show our model for diffracted waves in terms of elementary diffracted waves.

Assume $\bar{a} = \bar{a}/2$ in (16), that is, calculate $U^D$ at the boundary $B_0^c$. Then the following relations hold

$$p + n = -1, \quad (-\Phi/2 + \pi)^* = (-\Phi/2 - \pi)^*,$$

(17)

where (7) and (8) are used. Then inserting these relations into (16),

$$U^D(r, \theta) \bigg|_{\theta = \Phi/2} = 0, \quad \partial U^D(r, \theta) / \partial \theta \bigg|_{\theta = \Phi/2} = 0$$

(18)

is resulted for the soft wedge (m = -1) and

$$\partial U^D(r, \theta) / \partial \theta \bigg|_{\theta = \Phi/2} = 0$$

(19)

for the hard wedge (m=1). The same story holds for $U^D$ at the other boundary $B_0^c$. Consequently it becomes clear that the boundary conditions are satisfied in (15) and (16) inherently. It is the necessary condition that the expression for diffracted waves should satisfy but no conventional expressions have satisfied it so far.

### 4. Cell structure in diffracted waves

In the previous section the observation point $O$ is assumed to belong to $W_0$. The virtual space, however, can be constructed for $O$ that belongs to any $W_i$. Thus we can get the following relation

$$U^D(r, \theta + 2\pi) = -E(r, \theta + 3\pi) + E(r, \theta + \pi),$$

(20)

where $\bar{a}$ belongs to $W_p$ but $\bar{a} + 2\bar{a}$ doesn’t. From (15) and (20) the following relation holds

$$U^D(r, \theta) + U^D(r, \theta + 2\pi) = -E(r, \theta + 3\pi) + E(r, \theta - \pi).$$

(21)

Repeating this process for $N$ times we can get the following relation

$$\sum_{i=1}^N U^D(r, \theta + 2\pi(i - 1)) = -E(r, \theta + \pi(2N - 1)) + E(r, \theta - \pi),$$

(22)

where $N$ stands for a positive integer. Since $E$ is the periodic function of duration $2\bar{a}$, if the following relation holds

$$2N\pi = 2\Phi M,$$

(23)

where $M$ stands for a positive integer, (22) becomes
Thus if the aperture angle of wedge is given by
\[
\Phi = \frac{\pi N}{M},
\]
where \(N\) and \(M\) stand for relatively prime positive integers, the \(N\)-th order cell structure exists in the potential for diffracted waves in \(W_0\) and is characterized by (24). The characteristic of structure is unclear in (24) since it is expressed in terms of angle in the extended wedge region. Let make the characteristic clear by expressing (24) in terms of original angles. The problem of finding \(N\) original angles of
\[
\theta + 2\pi (i-1), \quad i = 1, 2, \ldots, N
\]
can be converted to that of (27) by making use of (11) and (25).
\[
\theta + 2\Phi (i-1)/N, \quad i = 1, 2, \ldots, N
\]
Note that there is no one by one coincidence between (26) and (27) except \(i = 1\). Then the original angles can be obtained by drawing \(N\) graphs, that is,
\[
y = f(x + 2(i-1)/N), \quad i = 1, 2, \ldots, N
\]
where the graph \(y = f(x)\) is shown in Figure 2(c) and the original angles are given by the intersections of the graphs with the vertical line \(x = \Phi / N\). This procedure is explained below using concrete examples.
\[ U^D (r, \theta) + U^D (r, \theta + 2\pi) = 0 \, , \]  
and it holds for the soft and hard wedges of aperture angle
\[ \Phi = 2\pi / M , \quad M = 1,3,5, \ldots \, , \]  
where \( M = 1 \) corresponds to a semi-infinite plane and \( M \) cannot take an even integer since \( N \) and \( M \) must be relatively prime. From (28) the graphs for original angle become
\[ y = f(x) = x \quad \text{and} \quad y = f(x + 1) = -x \, , \]
and are shown in Figure 4(a). Thus the original angle of \( \square + 2 \square \) in (31) is determined by the second relation in (33) as \( -\Phi \) and (31) is rewritten in terms of the original angles as
\[ U^D (r, \theta) + mU^D (r,-\theta) = 0 \, , \]  
where (9) is used and the change of slope in (33) means the change of the wedge number in the left-hand side of (34) as seen from Figure 2(b) and (c). In the case of hard wedge \( m = 1 \) (34) becomes
\[ U^D (r, \theta) = -U^D (r,-\theta) . \]  
This means that diffracted waves are antisymmetric with respect to \( \square = 0 \) and in the case of soft wedge \( m = -1 \) (34) becomes
\[ U^D (r, \theta) = U^D (r,-\theta) , \]  
and in this case diffracted waves are symmetric with respect to \( \square = 0 \). It has been observed that these properties exist in waves diffracted by the semi-infinite plane but its theoretical analysis becomes possible after the observation point is allowed to go outside \( W_0 \) as seen in (20). Let divide the wedge space into two cells \( C_1 \) and \( C_2 \) as shown in Figure 4(a). Then (35) and (36) can be regarded to show that the potential in one cell can be reconstructed by that in another cell. This idea is generalized in the next section.

4.3 The case of \( N \geq 3 \)
In the case of \( N = 3 \), (24) becomes
\[ U^D (r, \theta) + U^D (r, \theta + 2\pi) + U^D (r, \theta + 4\pi) = 0 , \]  
and it holds for the soft and hard wedges of aperture angle
\[ \Phi = 3\pi / M , \quad M = 2,4,5,7,8\ldots \ . \]  
From (28) the graphs for original angle become
\[ y = f(x) = x , \quad y = f(x + 2/3) \quad \text{and} \quad y = f(x + 4/3) \, , \]
and are shown in Figure 4(b). Let divide \( W_0 \) into \( N \) cells, that is, \( C_i (i = 1,2,\ldots,N) \) as
\[ C_i = \{ (r, \theta) | (-1/2 + (i - 1) / N)\Phi \leq \theta \leq (-1/2 + i / N)\Phi \} , \quad i = 1,2,\ldots,N \, , \]  
and \( \square_i \) be the angle of midpoint of \( C_i \) as
\[ \theta_i = (-1/2 + (2i - 1)/2N)\Phi , \quad i = 1,2,\ldots,N \ . \]  
Then a point in \( C \) can be specified by \( \square_i \) and the local coordinate \( \square \) as \( \square = \square_i + \square \) where \( \square_i \leq \square / 2N \). As seen in Figure 4 the intersections of \( x = (\square_i + 1) / \square \) with (39) are given by \( \square_p + (-1)^{p-i} \square , \quad p = 1,2,\ldots,N \). Then (24) can be rewritten in terms of original angles as
\[ \sum_{p = 1}^{N} m^{p-i} U^D (r, \theta_p + (-1)^{p-i} \alpha) = 0 \, , \]  
(42)
where $\mathbb{D} = \mathbb{D}_1 + \mathbb{D}_2$ is assumed. Thus the potential of diffracted waves in a cell $C_i$ can be expressed as

$$U^D(r, \theta + \alpha) = -\sum_{p=1}^{N} m^{p-i} U^D(r, \theta_p + (-1)^{p-i} \alpha),$$

(43)

where the potentials in $N-1$ cells remained are used in the right-hand side of (43). It is clear that (29), (35) and (36) are included in (43).

Figure 5 Graphs of potential for diffracted waves in a wedge of $\mathbb{D} = 3 \mathbb{D}_1/2$ where $r = 10 \mathbb{D}_1$, $r_S = 12 \mathbb{D}_1$ and $r_S = \mathbb{D}_1/4$. (a) and (c) correspond to the hard wedge and (b) and (d) to the soft wedge. Absolute values of sum of three potentials in $C^2$ are shown in (e).

4.4 Numerical results

The rigorous solutions of the potential in the hard and soft wedges are given by

$$U(r, \theta | m = 1) = -\frac{j \pi}{2 \Phi} \sum_{p=0}^{\infty} c_p J_{pv}(kr) H_{pv}^{(2)}(kr_S) \cos(pv(\theta + \Phi / 2)) \cos(pv(\theta_S + \Phi / 2)),$$

$$U(r, \theta | m = -1) = -\frac{j \pi}{2 \Phi} \sum_{p=0}^{\infty} c_p J_{pv}(kr) H_{pv}^{(2)}(kr_S) \sin(pv(\theta + \Phi / 2)) \sin(pv(\theta_S + \Phi / 2)),$$

(44)

respectively where $v = \mathbb{D}_1 / \mathbb{D}_2$, $J_{pv}$ and $H_{pv}^{(2)}$ stand for the Bessel function of real order and the second kind Hankel function of real order respectively, $\mathbb{D}_0 = 1$ and $\mathbb{D}_p = 2 (p \neq 0)$, and $r < r_S$ is assumed. By subtracting the geometrical optics solution from (44) the rigorous solution of $U^D$ is obtained.
The graphs of $U^D$ for $\Omega = 3 \pi/2$, $r = 10 \pi$, $r_3 = 12 \pi$ and $\Omega_3 = \pi/4$ are shown in Figure 5 where (a) and (c) correspond to the hard wedge ($m = 1$) and (b) and (d) to the soft wedge ($m = -1$). The potentials in these graphs are normalized by that of the direct waves at the shadow boundary and the real and imaginary parts of the normalized potential are shown in these figures. Consequently only the real part shows the jump of amplitude 1.0 at the shadow boundary at $x = -5/12$ and 1/12 where $x = \Omega / \pi$. Since the wedge belongs to the case of $N = 3$, the wedge is divided into three cells, that is, $C^D (-1/2 \leq x \leq -1/6)$, $C^S (-1/6 \leq x \leq 1/6)$ and $C^S (1/6 \leq x \leq 1/2)$. The curves in $C^D$ and $C^S$ that are folded into $C^S$ are shown by red lines. Note that in the case of soft wedge the polarity of folded potential is inverted as seen in Figure 5 (b) and (d). The absolute values of sum of three potentials in $C^S$ are shown in Figure (e). They are deviated from zero slightly that it can be claimed in Figure 5 (a) – (d) that three curves in $C^S$ are canceled each other. The same result will be obtained if curves are folded into $C^D$ or $C^S$. Thus the potential in a cell can be reconstructed by that in cells remained.

5 . Summary

If the aperture angle of wedge is expressed by $\Omega = N/M$ where $N$ and $M$ are relatively prime integers, diffracted waves in the wedge region can be divided equally into $N$ cells and the potentials in cells are cancelled each other after the cells are folded like a folding fan. In the case of soft wedges the amplitude is inverted in polarity at each folding. In other words the potential in a cell can be reconstructed by the fan-like sum of the potentials in $N-1$ cells remained. In the case of $N=1$ and 2, the cell structure is visually recognizable but in most cases it isn’t. Consequently it has been difficult to notice the structure without a proper mathematical model for diffracted waves.

In the previous reports the local property of waves diffracted by the wedge, that is, high-frequency approximate solution is studied using VDPD with considerable success \(^2,3\). And in this paper it is demonstrated that the new model works effectively in analyzing the global property of waves diffracted by the wedge. Thus the cell structure would be served as another strong support for VDPD.

References